Optimal Control

Control problem stated in terms of finding the control input for a given system which minimizes some objective function $J$.

Function minimization
- Unconstrained
- Constrained
  - Equality constraints
  - Inequality constraints

Expand function using Taylor series
$$f(x + \Delta x) = f(x) + g^T(x)\Delta x + \frac{1}{2}\Delta x^T H(x)\Delta x + \ldots$$

Want to find a minimum.

Optimal Control Problem

The problem statement is
$$\begin{align*}
\text{Minimize} & \quad J = h + \int_t^{t_f} g dt \\
\text{subject to } & \quad \dot{x} = f
\end{align*}$$

Ways to solve
- Direct transcription
- Variational approach
- Dynamic programming
  - DP problems usually have to be solved numerically
  - Basic recursion of DP basically says to start at the end of time and work backwards, at each step minimizing the cost just to get you to the prior step
  - Considering infinite horizon LQR (discrete first) the result is the Algebraic Riccati Equation and a constant static gain feedback control law
  - DP in continuous time

Principle of optimality—every sub-trajectory of an optimal trajectory is optimal.

Dynamic Programming

Continuous Time

Start with same problem statement, break limits of integration into two intervals, from $t$ to $t + \Delta t$ and then from $t + \Delta t$ to $t_f$. Use principle of optimality and express optimal cost as the cost to get from $t$ to $t + \Delta t$ and the cost-to-go to get from $t + \Delta t$ to $t_f$. Expand the cost-to-go using a Taylor series expansion, and approximate the over the integration interval $\Delta t$ by assuming $\Delta t$ is small. After some algebra, the result is a PDE known as the Hamilton-Jacobi-Bellman (HJB) equation. We simplify the notation in the HJB equation by defining the Hamiltonian.

$$-J^*_t = \min_u \mathcal{H}$$

$$\mathcal{H} = g + J^*_t f$$

LQR Using Dynamic Programming

Use the HJB equation. Write out the Hamiltonian and minimize with respect to $u$ to find $u^*$. Propose a form for $J^*$ plugging in $u^*$ to derive differential Riccati equation.

Variational Approach

Augment cost function with a Lagrange multiplier to enforce system dynamics. This Lagrange multiplier must be time varying, and we call the Lagrange multiplier the costate $p$. When we calculate the first variation and set to zero, a condition comes out specifying $\dot{p}$. Combining this differential equation with the one for the system dynamics we get a system of differential equations in $x$ and $p$, expressed with the Hamiltonian matrix $\bar{H}$. To solve, we guess a form of the solution $p$ and check that it works. The result is the Riccati differential equation. For infinite time, this differential equation reaches steady-state and becomes an algebraic equation. We can solve it to obtain the optimal control law, and thus the closed-loop system poles. We see that these are the poles (positive and negative) of $\bar{H}$ by doing a similarity transform on $\bar{H}$.

Optimal Control Note Sheet

Variational Approach to Optimal Control

Minimize $J = h + \int_t^{t_f} g dt$
subject to $\dot{x} = f$

Hamiltonian:
$$\mathcal{H} = g + p^T f$$

Necessary Conditions:
$$\dot{x} = H_p^T = f$$
$$\dot{p} = -H_x$$
$$H_u = 0$$

1. Write out Hamiltonian $\mathcal{H}$
2. Evaluate $H_u$ and $H_x$ to get costate equation and $u^*$
3. Can put state and costate equations into matrix form and solve eigenvalue problem using the Hamiltonian matrix $\mathcal{H}$.
   Solution is:
   $$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = v_1 e^{\lambda_1 t} + v_2 e^{\lambda_2 t}$$

4. Or can propose a solution $p = Px$ to get Riccati equation
5. If $t_f = \infty$ then $\bar{P} = 0$ in the Riccati equation

For LQR problem the Hamiltonian is
$$\mathcal{H} = \frac{1}{2} \begin{bmatrix} x^T R_{xx} x + u^T R_{uu} u \end{bmatrix} + p^T \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$
evaluate $H_x$ and $H_u$ we get
$$\mathcal{H}_x = x^T R_{xx} + p^T A$$
$$\mathcal{H}_u = u^T R_{uu} + p^T B$$

Plugging in
$$\ddot{p} = -R_{xx} x - A^T p$$
$$0 = R_{uu} u + B^T p$$

Solving the second condition for $u^*$
$$u^* = -R_{uu}^{-1} B^T p$$

Plugging in control we get following differential equations in matrix form
$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR_{uu}^{-1} B^T \\ -R_{xx} & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

To solve the state and costate equations, guess a solution
$$p = Px$$

Differentiate and simplify to get Riccati equation.
$$\hat{P} = PA + A^T P + R_{xx} - PBR_{uu}^{-1} B^T P$$

When we have infinite time horizon, $\hat{P} = 0$. We need to solve this equation to find $P$, and then use $p = Px$ with the $P$ we found to determine the control law $u$. For scalar systems we can easily solve the Riccati equation by hand. We can also compute the solution $P$ using the eigenvalues and eigenvectors of $\bar{H}$, although this approach doesn’t always work.
Constrained Optimization with Pontryagin

Minimize \[ J = h + \int_t^{t_f} g dt \]
subject to \[ \dot{x} = f \]
\[ c(x, u, t) \leq 0 \]

Hamiltonian: \[ \mathcal{H} = g + p^T f \]
Necessary Conditions:

| \[ \dot{x} = \mathcal{H}_p^T = f \] |
| \[ \dot{p} = -\mathcal{H}_x^T \] |
| \[ u^* = \arg \min_u \mathcal{H} \] |

1. Form Hamiltonian
2. Find \( u^* \) as it depends on sign of other stuff
   - Usually this stuff has \( p \) in it, so have to find costate equation
3. Plug each option for \( u^* \) into the state equations and integrate back to find expression for \( x(t) \)
   - Will use initial conditions for constants of integration
   - Can see switching curve for each option by plugging in final condition and eliminating \( t_f \)
4. More notes: \( h_x = p(t_f) \)

Solving the HJB Equation

Minimize \[ J = h + \int_t^{t_f} g dt \]
subject to \[ \dot{x} = f \]

Using dynamic programming in continuous time leads to the HJB equation.

Hamiltonian: \[ \mathcal{H} = g + J_x^* f \]
HJB Equation: \[ -J_t^* = \min_u \mathcal{H} \]
Boundary condition: \[ J^*(t_f) = h \]

1. Write out Hamiltonian \( \mathcal{H} \)
2. Evaluate \( \mathcal{H}_u = 0 \) and solve for \( u^* \) in terms of \( J_x^* \)
   - Can set \( \mathcal{H}_u = 0 \) since control is not constrained
3. Substitute \( u^* \) into HJB equation and simplify
4. Propose (guess) a solution form for \( J^* = px^2 \)
5. Plug solution into HJB
6. Solve for \( p(t) \), which should give \( J_x^* \) and thus \( u^* \)
   - Use boundary condition \( J^*(t_f) = h \)

For singular arc must have \( \mathcal{H}_u = 0 \) and \( \dot{\mathcal{H}}_u = 0 \). Remember in Goddard problem we dot \( \mathcal{H} = 0 \) to constrain \( u! \)

lec 18: For min time problems if system is completely controllable by \( B \), \( u \) can have no singular arcs. If not completely controllable \( u \) must have a singular interval.

For min time-fuel, if completely controllable by \( B \) and \( A \) nonsingular, there can be no singular intervals.

LQR vs KE \( P, Q, A, A^T ; B, C_x^T ; Q, R_{ww}; R, R_{vv} \)
LQG Stuff

Plant:  
\[ \dot{x} = Ax + Bu + B_w w \]
\[ y = Cx + v \]

Estimator: 
\[ \dot{x} = Ax + Bu + L(y - C\hat{x}) \]

Error: 
\[ \hat{x} = (A - LC)\hat{x} + B_w w - Lv \]

\( Q \) is the covariance matrix. Consider the following
\[ \dot{x} = Ax + B_w w \]
\[ E[\dot{x}(t)\hat{x}^\top(t)] = Q(t) \]
\[ \dot{Q} = E[(\dot{x}\hat{x}^\top + x\dot{x}^\top)] \]

This is the differential Lyapunov equation, which we substitute in for our terms in the estimation error equation and get:
\[ \dot{Q} = (A - LC)Q + Q(A - LC)^\top + B_w R_w + B_w^\top + LR_{uv} L^\top \]

Now the goal is:

- minimize \( J = E[\dot{x}(t_f)P_f\dot{x}(t_f)] \)
- subject to Differential Lyapunov equation

We can express \( J \) in terms of \( Q \) as
\[ J = \text{tr}\{E[P_f\dot{x}(t_f)\hat{x}^\top(t_f)]\} \]
\[ = \text{tr}\{P_fQ(t_f)\} \]

So make \( Q(t) \) the state \( L \) the control and its like an optimal control problem. Define Hamiltonian
\[ H = \text{tr}\{P\{A - LC\}Q + Q(A - LC)^\top + B_w R_w + B_w^\top + LR_{uv} L^\top\} \]

Optimal control satisfies
\[ H_L = 0 \]
\[ H_L = -PQC^\top + PLR_{uv} \]
\[ L = QC^\top R_{uv}^{-1} \]

Plugging \( L \) back into differential Lyapunov equation, we get estimation Riccati equation.
\[ \dot{Q} = AQ + QA^\top + B_w R_{ww} B_w^\top - QC^\top R_{uv}^{-1} CQ \]

More Notes

When \( \rho \to \infty \) (expensive) closed-loop poles are stable OL poles and reflection of unstable OL poles. \( \rho \to 0 \) control is cheap, so CL poles go to OL zeros (stable ones plus reflection of unstable) and remaining go to infinity. When control is cheap CL poles go to zeros and we minimize \( J \) by making things unobservable. Non-minimum phase zeros place fundamental limitation on cost.

If \( g \) and \( h \) not functions of time, \( H \) is constant along optimal trajectory.

Singular arcs \( H_u = 0 \) and \( \dot{H}_u = 0 \)

No cost along a singular arc.

To find optimal control for a continuous time system, can use Hamiltonian like always, write state and costate equations and put in matrix form. If it is infinite time, can solve for the the costate equation by \( P = V_2 V_1^{-\frac{1}{2}} e \). (see lec 11–3).

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / \det \]

\[ \frac{d}{dt} \text{tr}(AXB) = A^\top B^\top \]

Optimal Control Note Sheet

If \( A \) is invertible, then
\[ \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det(D - CA^{-1}B) \]

\( A \) and \( B \) square \( \det(AB) = \det(A)\det(B) \). For any matrices \( C \) and \( D \) such that \( CD \) is square, \( \det(I + CD) = \det(I + DC) \). Symmetric root locus
\[ \tilde{H} = \begin{bmatrix} A & -BR_{uu}^{-1}B^\top \\ -R_{xx} & -A^\top \end{bmatrix} \]

\[ \det(sI - \tilde{H}) = (-1)^n \{ D(s)D(-s) + \frac{R_{zz}}{R_{uu}} N(s)N(-s) \} = 0 \]

where \( \frac{R_{zz}}{R_{uu}} = \frac{1}{p} \), \( G_{uu}(s) = C_z(sI-A)^{-1}B_u \) and \( D(s) = \det(sI - A) \). Use the 180 degree root locus for \( n - m \) even, 0 degree root locus for \( n - m \) odd. 180 degree is the “normal” root locus.

To plot SRL given \( (A, B, C_z) \). Write TF from \( u \) to \( z \). Plot all poles and zeros. Reflect all poles and zeros across imaginary axis. Check \( n - m \) and draw asymptotes.

LQR Margins

Loop transfer function from \( u \) to \( -u \)
\[ L(s) = K(sI_a)^{-1} \]

Kalman frequency domain equality
\[ [I + L(-s)]^{-1} [I + L(s)] = I + \frac{1}{\rho} G_{uu}(s) \]

lec 14: For free final time problems \( h_t + H(t_f) = 0 \)

lec 15: Augmenting Hamiltonian with point constraint. \( J_a = h + \int_a^b H - p^\top f dt + \pi^\top N \) we find \( p^\top (t_1^+) = p^\top (t_1^-) - \pi^\top N_x(t_1) \) and \( H(t_1^+) = H(t_1^-) + \pi^\top N_t(t_1) \)

lec 17: WHEN \( t_f \) AND \( x(t_f) \) ARE FREE TRANSVERSALITY IS:
\[ h(t_f) + H(t_f) = 0 \]

(this might be true even when \( x(t_f) \) is fixed)

Constrained Optimization with Augmented Hamiltonian

Minimize \( J = h + \int_{t_0}^{t_f} g dt \)
subject to
\[ \dot{x} = f \]
\[ c(x, u, t) \leq 0 \]

Hamiltonian:
\[ H = g + p^\top f \]

Aug. Hamiltonian:
\[ H_a = H + \nu^\top c \]

Necessary Conditions:
\[ \dot{p} = -(H_a)^\top \]
\[ (H_a)_x = 0 \]
\[ \nu_t \geq 0 \ 	ext{if} \ c_t = 0 \ \text{(active)} \]
\[ \nu_t \leq 0 \ 	ext{if} \ c_t \leq 0 \ \text{(inactive)} \]
General Stuff

Taylor series expansion

\[ f(x + \Delta x) = f(x) + g^\top(x)\Delta x + \frac{1}{2}\Delta x^\top H(x)\Delta x + \ldots \]
where \( g(x) = \nabla f(x) \) is the gradient and \( H(x) \) is the Hessian

\[
g(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \ldots & \frac{\partial f}{\partial x_n} \end{bmatrix}^\top
\]

\[
H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\
\frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \ldots & \frac{\partial^2 f}{\partial x_2\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_n^2} 
\end{bmatrix}
\]

Stationary point \( g(x^*) = 0 \). If \( H(x^*) \succ 0 \) then \( x^* \) is a strong (weak) minimum.

Newton’s Method

Guess a stationary point \( x_k \). Then can write actual stationary point as \( x = x_k + \Delta x_k \) and we have \( f(x) = f(x_k + \Delta x_k) \approx f(x_k) + g(x_k)^\top \Delta x_k + \frac{1}{2} \Delta x_k^\top H(x_k) \Delta x_k \) and we have \( \nabla f(x) \approx g(x_k) + H(x_k)\Delta x_k \). And we want this to be zero so set it to zero and get \( x_{k+1} = x_k - H_k^{-1}g_k \).

Convergence Rates

Constrained Minimization

Equality Constraints

Given a problem in the form

\[
\text{optimize } f(x, y) \\
\text{subject to } c(x, y) = 0
\]

the following approach will find stationary points which satisfy the constraint.

1. Form Lagrangian

\[
\text{Lagrangian: } L(x, y, \lambda) = f(x, y) + \lambda^\top c(x, y)
\]

2. Form necessary conditions

\[
\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial \lambda} = 0
\]

3. Solve and find the solution (if there are more than one) that corresponds to either min or max

Inequality Constraints

Given a problem in the form

\[
\text{optimize } f(x, y) \\
\text{subject to } c_1(x, y) \leq 0 \\
\quad c_2(x, y) \geq 0
\]

Approach is similar to equality constraints. Only this time, we have different cases, \( \lambda_1 = 0 \) meaning the constraint is inactive, or \( \lambda_1 \neq 0 \) means the constraint is active.

1. Rewrite the inequality constraints so they are all in the form less than zero

\[
c_i(x, y) \leq 0
\]

2. Form Lagrangian as: \( L(x, y, \lambda) = f(x, y) + \lambda^\top c(x, y) \)

3. Form necessary conditions (note change from before)

\[
\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial y} = 0 \quad \lambda_i \frac{\partial L}{\partial \lambda_i} = 0
\]

4. Consider different cases whether each constraint is active/inactive

(a) Both constraints inactive \( \lambda_1 = \lambda_2 = 0 \). Plug this into equations, solve, verify the constraints are satisfied.

(b) Inactive: \( \lambda_1 = 0 \), active: \( \lambda_2 \neq 0 \). Same as the above.

Optimal Control Note Sheet

5. Look at each case above and evaluate \( f(x, y) \) at each stationary point which satisfies the constraints to find the min/max.

Note: according to KKT conditions, all active inequality constraints must have \( \lambda_i \geq 0 \), if not it is not a valid solution

Constraint Sensitivity

To find the sensitivity of a cost to a constraint, write the constraint, say \( c_1(x, y) \leq 0 \) as \( c_1(x, y) \leq C_1 \). Then we calculate the change in cost as follows, where say the constraint \( C_1 \) is changed to \( C_1 + \Delta C_1 \).

\[
\Delta f = -\lambda_1 \Delta C_1
\]

Barrier Functions.

Adjoint Methods

Computer program \( y = f(x) \) where \( y \) is a scalar cost and \( x \) is a vector. We want to evaluate the gradient, \( g(x) = \frac{\partial y}{\partial x} \). If in the computer program we calculate \( y \) by operating on \( x \) throughout the code as \( z_1 = F_1(x_1) \) then \( x_2 = F_2(z_1) \) and so on we have \( y = F_n(F_{n-1}(F_{n-2}(\ldots x\ldots))) \) and so to get the gradient it is lots of chain rules. \( \frac{\partial y}{\partial x} = \frac{\partial F_n}{\partial z_{n-1}} \frac{\partial z_{n-1}}{\partial z_{n-2}} \ldots \frac{\partial F_1}{\partial x} \). Depending on how we group these multiplications is either forward or reverse accumulation. Reverse is better because we the number of variables to do each operation is less \( (1 \times m) \) instead of \( (m \times n) \) with forward.

Example 4–13

\[
y = f(x_1, x_2) = \sin x_1 + x_1 x_2. \quad \text{Forward code: (1) } t_1 = \sin x_1, \quad (2) t_2 = x_1 t_1, \quad (3) y = t_1 t_2. \quad \text{Reverse code: form } z_{n-1} = F_n(z_{n-1}).
\]

Dynamic Programming

Discrete Time

The problem statement is

\[
\text{Minimize } J(h(x_N, N) + \sum_{k=0}^{N-1} g_k(x_k, u_k))
\]

subject to \( x_{k+1} = f_k(x_k, u_k) \)

with \( x_0 \) given and \( u \in \Omega \) feasible controls. We break this up by defining the cost-to-go, which is the cost to get to any point \( k \) to the end as

\[
V_k(x) = \min_{u_k} \left\{ h(x_N, N) + \sum_{i=k}^{N-1} g_i(x_i, u_i) \right\}
\]

We can break the cost-to-go up and get the basic recursion of dynamic programming

\[
V_k(x) = \min_{u_k} \left\{ g_k(x_k, k) + V_{k+1}(f(x, u_k)) \right\}
\]
Discrete LQR

Analytical solutions don’t exist for many DP problems, but this is not the case for LQR.

Minimize \[ J = \frac{1}{2} x_N^T H_N x_N + \sum_{k=0}^{N-1} \frac{1}{2} x_k^T Q_k x_k + u_k^T R_k u_k \]
subject to \[ x_{k+1} = A_k x_k + B_k u_k \]
where \( H > 0, Q \geq 0, R > 0 \). To solve use the basic recursion of dynamic programming.

1. Evaluate \( V_{N-1} \) using given \( h(x_N) \) and \( g_k(x_k, u_k) \).
2. Plug in dynamics where \( x_N = A_{N-1} x_{N-1} + B_{N-1} u_{N-1} \).
3. We want to find the control to minimize \( V_{N-1} \), so differentiate the right hand side to get the following, which we will transpose and then set to zero
\[ \frac{\partial V_{N-1}}{\partial u_{N-1}} = u_{N-1} R_{N-1} + \{ A x + B u \}_{N-1} H B_{N-1} = 0 \]
so doing algebra we get
\[ u_{N-1}^* = - \left( R_{N-1} + B_{N-1}^T H B_{N-1} \right)^{-1} B_{N-1}^T H A_{N-1} x_{N-1} \]

4. Now plug optimal control \( u_{N-1}^* \) into \( V_{N-1} \) to get
\[ V_{N-1} = \frac{1}{2} x_{N-1}^T \left( Q + F^T R F + (A - B F)^T H (A - B F) \right) x_{N-1} - \sum_{k=N-1}^{P_{N-1}} \frac{1}{2} x_k^T P_k x_k \]

5. Looking at the above expression, we now assume a solution \( V_k(x_k) = \frac{1}{2} x_k^T P_k x_k \) and then find \( V_{k-1}(x_{k-1}) = \frac{1}{2} x_{k-1}^T P_{k-1} x_{k-1} \). Note that \( P_N = H \).
\[ P_k = \frac{1}{2} x_k x_k + \left( A_k - B_k F_{k-1} \right)^T P_{k-1} \left( A_k - B_k F_{k-1} \right) - \frac{1}{2} x_{k-1}^T P_k x_{k-1} \]

6. Control is \( u_k = -F_k x_k \). \( P_k \) and \( F_k \) are independent of the state and can be computed off-line. If we plug the expression for \( F_k \) into \( P_k \) we get discrete time Riccati equation.
7. If the system is LTI, \( A, B, Q, R \) are constants and so replace \( P \) with \( P_{ss} \) and solve Algebraic Riccati Equation.
\[ P_{ss} = Q + A^T \left( P_{ss} - P_{ss} B (R + B^T P_{ss} B)^{-1} B^T P_{ss} A \right) \]

Continuous Time

The problem statement is

Minimize \( J = h(x(t), t_f) + \int_t^{t_f} g(x(t), u(t), t) dt \)
subject to \( \dot{x} = f(x, u) \)

Instead of breaking the sum up in the cost we go, break the integral up from \( t \) to \( t + \Delta t \) and \( t + \Delta t \) to \( t_f \).

\[ J^*(x(t), t) = \min_u \left\{ J_{t}^{t_f} g(x, u, \tau) d\tau + J^*(x(t + \Delta t), t + \Delta t) \right\} \]

1. Represent the \( J^*(x(t + \Delta t), t + \Delta t) \) term using Taylor series.
\[ J^*(x(t + \Delta t), t + \Delta t) \approx J^*(x(t), t) + J^*_x(x(t), t) \Delta t + J^*_t(x(t), t) \frac{\Delta t}{2} \]

2. Must ensure first order terms match.
\[ -J^*_x(x, t) = \min_u \mathcal{H} \]

where
\[ \mathcal{H} = g(x, u, t) + J_{x}^* (x, t) f(x, u, t) \]

To solve CT LQR guess \( J^* = \frac{1}{2} x^T(t) P(t) x(t) \), form Hamiltonian, then minimize \( \mathcal{H} \) wrt to \( u \). So take derivation and get \( u^* = -R_{uu}^{-1}(t) B^T(t) P(t) x(t) \). Plug this control back into \( H \) and then to satisfy HJB we must have \( -J^*_t = -\frac{1}{2} x^T(t) P(t) x(t) \), giving differential Riccati equation.

Optimal Control Note Sheet

Calculus of Variations

Derivation of Euler-Lagrange Equation

We find increment defined as \( \Delta J = J(x + \delta x) - J(x) \). Then simplify this increment to first order to get the variation, which by the FToCOV says \( \delta J(x^*(t), \delta x(t)) = 0 \). Start with cost
\[ J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt \]

Evaluate perturbed cost
\[ J(x + \delta x) = \int_{t_0}^{t_f+\delta t} g(x + \delta x, \dot{x} + \delta \dot{x}, t) dt \]

1. Split \( J(x + \delta x) \) integral into three pieces
2. Approximate the integrals using \( \int_{t_0}^{t_f+\delta t} f(\tau) d\tau \approx f(t) \delta t \)
3. Approximate integrand using Taylor series
\[ g(x + \delta x, \dot{x} + \delta \dot{x}, t) \approx g(x, \dot{x}, t) + \frac{\partial g}{\partial x} \delta x + \frac{\partial g}{\partial \dot{x}} \delta \dot{x} \]

4. Neglect terms higher than first order, and then subtract \( \delta J = J(x + \delta x) - J(x) \) and simplify

Integrate by parts
\[ u = \frac{\partial g}{\partial x}, \quad dv = \frac{d}{dt} \frac{\partial g}{\partial \dot{x}} dt, \quad \delta x dt = \frac{d}{dx} \frac{\partial g}{\partial \dot{x}} \delta \dot{x} dt \]

6. Simplify using following at \( t_0 \) and \( t_f \)
\[ \delta x(t_0) \approx \delta x_0 \]

By setting this to zero for arbitrary \( \delta x \), and by applying constraints e.g. \( t_0 \) fixed, \( x(t_f) \) free, we get the necessary conditions.

General Terminal Conditions

If there are general terminal conditions \( m(x(t_f), t_f) = 0 \) augment cost with them using Lagrange multiplier \( \nu \). If cost is
\[ J(x) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt \]

augment as
\[ J_\nu(x) = h(x(t_f), t_f) + \nu^T m(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, \dot{x}, t) dt \]

Do same thing as before to get necessary conditions: \( g_x - \frac{\partial}{\partial x} g_x = 0 \), \( g_x + g_{\dot{x}} \dot{x} \bigg|_{t_f} = 0 \), \( w_{x_{t_f}} + g - g_{\dot{x}} \dot{x} \bigg|_{t_f} = 0 \), and \( m(x(t_f), t_f) = 0 \).

Variational Approach to Optimal Control

Augment cost using Lagrange multiplier \( p(t) \) as:
\[ J_\nu(x) = h(x(t_f), t_f) + \int_{t_0}^{t_f} \left\{ g(x, u, t) + p^T (f(x, u, t) - \dot{x}) \right\} dt \]

\[ \mathcal{H}(x, u, p, t) = g(x, u, t) + p^T f(x, u, t) \]

\[ w(x(t_f), u, t_f) = h(x(t_f), t_f) + \nu^T m(x(t_f), t_f) \]

**Necessary Conditions:**
\[ \dot{x} = H^p = f(x, u, t) \]
\[ \dot{p} = -H^x \]
\[ H_u = 0 \]

Boundary conditions: if \( t_f \) is not fixed use \( H(t_f) + w_{t_f}(t_f) = 0 \), otherwise don’t use it. But use these regardless: \( m(x(t_f), t_f) = 0 \), \( x(t_0), t_0 \) given, \( p(t_f) = \left( \frac{\partial w}{\partial x} \right)^T \).