Single Particle

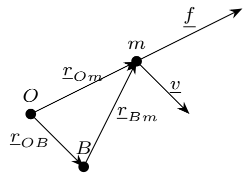

Figure 1: **Point mass m under action of force $\underline{f}$. Point O is fixed in inertial space, and point B is a general point, not necessarily fixed in inertial space.**

$$\underline{p} = m\underline{v}$$

$$\underline{f} = \frac{d\underline{p}}{dt}$$

$$\tau_O = \underline{r}_{Om} \times \underline{f}$$

$$\boxed{\underline{h}_O = \underline{r}_{Om} \times \underline{p}}$$

$$\underline{\tau}_O = \frac{d}{dt}\underline{h}_O$$

$$\underline{\tau}_B = \underline{\tau}_O - \underline{r}_{OB} \times \underline{f}$$

$$\underline{h}_B = \underline{h}_O - \underline{r}_{OB} \times \underline{p}$$

$$\underline{\tau}_B = \frac{d}{dt}\underline{h}_B + \underline{v}_B \times \underline{p}$$

Kinematics of Rigid Bodies

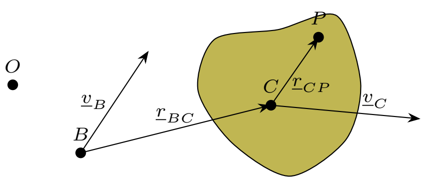

Figure 2: **Rigid body. O is inertially fixed in space, and point B is a general point, which can be moving, about which we may take moments. Point C is the body center of mass, and point P is some other point fixed on the body**

Note that the vector notation r_{BC} means a vector from point B to point C.

$$\boxed{\underline{P} = m\underline{v}_C}$$

$$\underline{\tau}_O^{\text{ext}} = \frac{d}{dt}\underline{H}_O$$

$$\boxed{\underline{\tau}_B^{\text{ext}} = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}}$$

$$\underline{H}_B = [I]_B\omega$$

$$\boxed{\underline{F}^{\text{ext}} = \frac{d}{dt}\underline{P}}$$

$$\boxed{\underline{H}_B = \underline{H}_C + \underline{r}_{BC} \times \underline{P}}$$

$$\underline{H}_B = \underline{H}_O + \underline{r}_{BO} \times \underline{P}$$

$$\boxed{\underline{v}_P = \underline{v}_C + \underline{\omega} \times \underline{r}_{CP}}$$

For non-holonomic (rolling coin): $$\underline{\tau}_B^{\text{ext}} = \frac{d}{dt}\underline{H}_C + \underline{r}_{BC} \times \frac{d}{dt}\underline{P}$$

To find principal axes

$$I_{\text{principal}} - \lambda I = 0$$

Impulse

Take linear and angular momentum principle for a rigid body

$$\underline{F}^{\text{ext}} = \frac{d}{dt}\underline{P}$$

$$\underline{\tau}_B^{\text{ext}} = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$$

and separate them and integrate them over a short period of time

$$\int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt = \int_{\underline{P}(0^-)}^{\underline{P}(0^+)} d\underline{P}$$

$$\int_{t=0^-}^{t=0^+} \underline{\tau}_B^{\text{ext}} dt = \int_{\underline{H}_B(0^-)}^{\underline{H}_B(0^+)} d\underline{H}_B + \int_{t=0^-}^{t=0^+} \underline{v}_B \times \underline{P} dt$$

$$\Delta \underline{P} = \int_{t=0^-}^{t=0^+} \underline{F}^{\text{ext}} dt$$

Remember when integrating from $t = 0^-$ to $t = 0^+$ constant forces like gravity integrate to zero

Work and Energy Principles

KE of rigid body rotating about CM:

$$T = \frac{1}{2}Mv_G^2 + \frac{1}{2}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)$$

Finding Center of Mass and Moment of Inertia

P is a general point.

Parallel axis theorem: $$I_P = I_C + Mh^2$$

$$[I]_P = [I]_C + M\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

Moments of inertia $$I_x = \int_m (y^2+z^2)dm$$

Products of inertia $$I_{xy} = \int_V \rho xy dV$$

$$I_x = \rho \int_V (y^2+z^2)dV$$

x CM: $$x_{cm} = \frac{\sum_i A_i r_i}{\sum_i A_i}$$

y CM: $$z_{cm} = \frac{\int_m z dm}{\int_m dm}$$

z CM: $$z_{cm} = \frac{\int_V z dV}{\int_V dV}$$

Areas, Volumes, Centroids, Moments of Inertia

$$A_{\text{sphere}} = 4\pi r^2$$

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Through axis of rot. $$I_{\text{cylinder},z} = \frac{1}{2}mr^2$$

Through center $$I_{\text{cylinder},x,y} = \frac{1}{12}m(3r^2+h^2)$$

Rod length L about end: $$I_{\text{rod,end}} = \frac{1}{3}mL^2$$

Rod length L about center: $$I_{\text{rod,center}} = \frac{1}{12}mL^2$$

Sphere radius r: $$I_{\text{sphere}} = \frac{2}{5}mr^2$$

Cone $$I_{\text{cone},z} = \frac{3}{10}mr^2$$

Cube thru cent $l(x), w(y), h(z)$ $$I_{\text{cube},x} = \frac{1}{12}(w^2+h^2)$$

Axes at tip of cone $$I_{\text{cone},x,y} = \frac{3}{80}m(4r^2+h^2)$$

Axes at base of cone $$I_{\text{cone},x,y} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2)$$

through center of hoop $$I_{\text{hoop},z} = mr^2$$

Centroid of cone up from base $$z = \frac{h}{4}$$

Lagrange's Method to find EOM

1. Identify number of generalized coordinates and any generalized forces
2. Choose generalized coordinates $\xi_1, \xi_2, \ldots$
3. Find kinetic energy T and potential energy V in terms of these generalized coordinates
4. Assemble Lagrangian

$$\boxed{\mathscr{L} = T - V}$$

5. Express generalized forces in terms of the generalized coordinates

$$\delta W = F_x \delta x$$
$$\delta W = \Xi_j \delta \xi_j$$

- When finding generalized forces which require solving δx in terms of $\delta\xi$, sometimes it is easiest to find velocities, then cancel dt and make dx into δx and $d\xi$ into $\delta\xi$.
- When breaking force F into components F_x, don't forget the sign
- Measure springs deflections from static equilibrium and gravity won't appear in equations of motion
- When finding kinetic energy of rigid bodies, place coordinate system at CG and such that it is a set of principal axes, then the moment of inertia is *about the CG* not the physical point of rotation.

6. Evaluate

$$\frac{\partial \mathscr{L}}{\partial \dot{\xi}_j} \quad \text{and} \quad \frac{\partial \mathscr{L}}{\partial \xi_j}$$

7. Use the formula

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\xi}_j}\right) - \frac{\partial \mathscr{L}}{\partial \xi_j} = \Xi_j}$$

That gives us the equations of motion

Stability Analysis of Discrete Systems

1. Use Lagrange's method to get EOM
2. Identify steady motions
   - If ξ_j does not show up explicitly in the Lagrangian $\mathscr{L}$, it is *ignorable*, or *cyclic*. Set
   $$\dot{\xi}_{\text{ignorable}} = \text{constant}$$
   - If ξ_j does show up in the Lagrangian $\mathscr{L}$, it is *non-ignorable*. Set
   $$\xi_{\text{non-ignorable}} = \xi_s = \text{constant}$$
3. Linearize the equations of motion. The form is
   $$[M]\ddot{x} + [K]x = 0$$
   From here there are two options
4. Solve for the natural frequencies and mode shapes
   - Let $\dot{x} = sx$
   - Solve $\left([M]s^2 + [K]\right)x = 0 \quad \Rightarrow \quad \boxed{\det\left([K] - \omega_i^2[M]\right) = 0}$
   - This is an eigenvalue problem where the eigenvalues are the natural frequencies, and the eigenvectors are the mode shapes

   Alternatively *This way requires $[M]$ and $[K]$*
5. Guess as many modes $\{a\}_i$ as possible
6. Use orthogonality to verify guessed modes, and find new modes
   $$\boxed{\begin{aligned}\{a\}_i^\top[M]\{a\}_j &= 0\\ \{a\}_i^\top[K]\{a\}_j &= 0\end{aligned}}$$
   The orthogonality condition comes from left multiplying $\left([K] - \omega_i^2[M]\right)\{a\}_i = 0$ for two cases with i and j by two modes which are orthogonal, $\{a\}_j^\top$ and $\{a\}_i^\top$.
7. Use **Rayleigh quotient** to find ω_i
   $$\boxed{\omega_i^2 = \frac{\{a\}_i^\top[K]\{a\}_i}{\{a\}_i^\top[M]\{a\}_i}}$$
   which comes from $\left([K] - \omega_i^2[M]\right)\{a\}_i = 0$ and left multiplying by $\{a\}_i^\top$

For the system $[M]\{\ddot{x}\} + [K]\{x\} = \{F\}\sin\omega t$, after we find the modes, we can put the modes into a matrix $[\Phi]$ and use this matrix to come up with a new system with vector $\{u\}$, where the mass and spring matrix are diagonal. Let $\{x\} = [\Phi]\{u\}$. Plugging this in we get

$$\underbrace{[\Phi]^\top[M][\Phi]}_{[M]_D}\{\ddot{u}\} + \underbrace{[\Phi]^\top[K][\Phi]}_{[K]_D}\{u\} = [\Phi]^\top\{F\}\sin\omega t$$

Derivations for Continuous Systems

Wave Equation for a String

String with mass/length ρ under tension T. So mass of a little piece is $dm = \rho dx$. String has length s with angle α on left side, $\alpha + \frac{\partial\alpha}{\partial x}dx$ on the right side, and the angle is small, so the string is approximately length dx.

Conservation of momentum in x-direction: the string does not move in the x-direction

$$T(x+dx)\cos\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right) - T(x)\cos(\alpha) = 0$$

Expanding $\cos\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right)$

$$\cos\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right) = \cos(\alpha)\cos\left(\frac{\partial\alpha}{\partial x}dx\right) - \sin(\alpha)\sin\left(\frac{\partial\alpha}{\partial x}dx\right)$$

Substituting this in

$$T(x+dx)\left[\cos(\alpha)\cos\left(\frac{\partial\alpha}{\partial x}dx\right) - \sin(\alpha)\sin\left(\frac{\partial\alpha}{\partial x}dx\right)\right] - T(x)\cos(\alpha) = 0$$

Divide both sides by dx and take the limit as $dx \to 0$

$$\boxed{T(x) = T = \text{constant}}$$

Conservation of momentum in y-direction:

$$T(x+dx)\sin\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right) - T(x)\sin(\alpha) = \frac{d^2y}{dt^2}\rho dx$$

Expanding $\sin\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right)$

$$\sin\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right) = \sin(\alpha)\cos\left(\frac{\partial\alpha}{\partial x}dx\right) + \cos(\alpha)\sin\left(\frac{\partial\alpha}{\partial x}dx\right)$$

Plugging in, and using the fact that $T(x) = T$ we have

$$T\left[\sin(\alpha)\cos\left(\frac{\partial\alpha}{\partial x}dx\right) + \cos(\alpha)\sin\left(\frac{\partial\alpha}{\partial x}dx\right)\right] - T\sin(\alpha) = \frac{d^2y}{dt^2}\rho dx$$

Using small angle approximations we get

$$T\left[\sin(\alpha) + \frac{\partial\alpha}{\partial x}dx\right] - T\sin(\alpha) = \frac{d^2y}{dt^2}\rho dx$$

Simplifying, we get

$$T\frac{\partial\alpha}{\partial x} = \frac{d^2y}{dt^2}\rho$$

Using small angle assumption again where $\alpha \approx \tan(\alpha) = \frac{dy}{dx}$ we have

$$\boxed{T\frac{\partial^2 y}{\partial x^2} = \frac{d^2y}{dt^2}\rho}$$

Can add forcing as

$$\frac{d^2y}{dt^2}\rho = T\frac{\partial^2 y}{\partial x^2} + f(x,t)$$

Euler-Bernoulli Beam Equation

Beam with **mass/length** ρA, with **internal shear force** Q, **bending moment** M_b, **height** $y(x,t)$. Square piece of block with shear force Q down on left side, $Q + \frac{\partial Q}{\partial x}dx$ pointing up on the right side, and moments M_b going up on the left side and $M_b + \frac{\partial M_b}{\partial x}dx$ going up on the right side. The **constitutive law** for a bending beam relates moment to curvature as

$$\boxed{M_b = EI\frac{\partial^2 y}{\partial x^2}}$$

Conservation of angular momentum about right side gives

$$M_b + \frac{\partial M_b}{\partial x} - M_b + Qdx = 0$$

gives

$$\boxed{Q = -\frac{\partial M_b}{\partial x}}$$

Substituting the constitutive law

$$Q = -EI\frac{\partial^3 y}{\partial x^3}$$

Conservation of linear momentum in y-direction gives

$$(\rho A dx)\frac{\partial^2 y}{\partial t^2} = \frac{\partial Q}{\partial x}dx$$

Evaluating $\frac{\partial Q}{\partial x}$ using the expression for Q derived using conservation of angular momentum

$$\frac{\partial Q}{\partial x} = -EI\frac{\partial^4 y}{\partial x^4}$$

Substituting in

$$\boxed{\rho A\frac{\partial^2 y}{\partial t^2} = -EI\frac{\partial^4 y}{\partial x^4}}$$

Self adjoint means solution is separable.

Longitudinal Displacement (Stretching) of a Rod

$$\boxed{\rho A\frac{\partial^2 \xi}{\partial t^2} = EA\frac{\partial^2 \xi}{\partial x^2}}$$

Axial Displacement (Twisting) of a Shaft

$$\boxed{\rho J\frac{\partial^2 \phi}{\partial t^2} = GJ\frac{\partial^2 \phi}{\partial x^2}}$$

Solving Continuous Systems

Solving String Problems with Forcing

To solve the forced response, always solve the unforced problem first.

1. Write down governing equation
   $$\boxed{\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} + f(x,t)}$$
2. Propose a solution of the form
   $$y(x,t) = a(x)\cos(\omega t)$$
   where the time varying harmonic function matches that of the forcing function (including frequency)
3. Plut this solution in, and obtain a simplified ODE in $a(x)$
   $$-\rho\omega^2 a(x)\cos(\omega t) = T\frac{d^2a}{dx^2}\cos(\omega t) + f(x,t)$$
   Take for example the forcing function to be $f(x,t) = F_0\cos(\omega t)$ then
   $$-\rho\omega^2 a(x) = T\frac{d^2a}{dx^2} + F_0$$
   rearranging
   $$\frac{d^2a}{dx^2} + \frac{\rho\omega^2}{T}a(x) = -F_0$$
   using $\lambda^2 = \frac{\rho\omega^2}{T}$
   $$\frac{d^2a}{dx^2} + \lambda^2 a(x) = -F_0$$
4. Find the homogeneous solution $a_h(x)$ and particular solution $a_p(x)$. Propose the homogeneous solution
   $$a(x) = Ae^{Bx}$$
   which has second derivative
   $$\frac{d^2a}{dx^2} = B^2Ae^{Bx}$$
   plugging in we get
   $$B = \pm\lambda i$$
   So the solutions are
   $$a_1(x) = A_1e^{\lambda ix}$$
   $$a_2(x) = A_2e^{\lambda - ix}$$
   giving
   $$a_h(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x)$$

5. Solution to forced equation should be constant
6. Form the total solution by adding the homogeneous and particular solutions, and apply BCs

Answering the question

- Determine the steady-state vibration just means find $y(x,t)$
- Identifying resonances is to find values of ω where the solution blows up

Solving String Problems with a Mass on them

The governing equation for this is the same as a regular string, but the solution is not valid across mass. Will need to use two solutions, one valid on each side of the mass.

1. Write down governing equation
$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$$
2. Consider solving this equation between the left end at $x=-L$ and the mass, and then between the mass and the right end at $x=L$. The general form of the solution is
$$y_L(x,t) = a_L(x)\sin(\omega t)$$
$$y_R(x,t) = a_R(x)\sin(\omega t)$$
Leads to
$$y_L(x,t) = C_L\sin(\lambda x+\phi_L)\sin(\omega t)$$
$$y_R(x,t) = C_R\sin(\lambda x+\phi_R)\sin(\omega t)$$
where $\lambda^2 = \frac{\rho\omega^2}{T}$. Should get $\phi_L = \lambda L$ and $\phi_R = -\lambda L$
3. Apply 4 boundary conditions: each end of the string, and the matching conditions at the mass
$$\boxed{y_L(x_m,t) = y_R(x_m,t)}$$
and the following, which comes from **linear momentum of mass in y-direction**
$$\boxed{M\frac{\partial^2 y}{\partial t^2}\Big|_{x_m} = T\left(\frac{\partial y_R}{\partial x}\Big|_{x_m} - \frac{\partial y_L}{\partial x}\Big|_{x_m}\right)}$$
Where either $y_L(x,t)$ or $y_R(x,t)$ can be used to evaluate the second derivative. Use $y_R(x,t)$.
4. Arranging these two conditions in matrix form
$$\begin{bmatrix} -\sin(\lambda(x_m-L)) & \sin(\lambda(x_m+L)) \\ -M\lambda^2\sin(\lambda(x_m-L)) - & \\ T\lambda\cos(\lambda(x_m-L)) & -T\lambda\cos(\lambda(x_m-L)) \end{bmatrix}\begin{bmatrix} C_R \\ C_L\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
5. Special case is when the mass is in the middle of the string at x_m. This reduces the above by taking the determinant to
$$\sin(\lambda L)\left[M\lambda^2\sin(\lambda L) - 2T\lambda\cos(\lambda L)\right] = 0$$
This equation can be solved to find the frequencies ω_n from each λ_n.
6. Mode shapes are those when the mass is stationary

Solving Beam Problems

1. Write down governing equation
$$\boxed{\rho A\frac{\partial^2 y}{\partial t^2} = -EI\frac{\partial^4 y}{\partial x^4}}$$
2. Propose following solution solution for beam problems. Can show such a separable solution works.
$$\boxed{y(x,t) = a(x)\sin(\omega t)}$$
The second and fourth derivatives of general solution to beam equation are
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a(x)\sin(\omega t)$$
$$\frac{d^4 y}{dx^4} = \frac{d^4 a(x)}{dx^4}\sin(\omega t)$$
3. Plugging the proposed solution into the governing equation we get
$$\boxed{\frac{d^4 a}{dx^4} - \lambda^4 a(x) = 0} \quad \text{where} \quad \boxed{\lambda^4 = \frac{\rho A\omega^2}{EI}}$$
4. The general solution to this ODE is
$$\boxed{a(x) = C_1\sin\lambda x + C_2\cos\lambda x + C_3\sinh\lambda x + C_4\cosh\lambda x}$$
5. Apply boundary conditions to solve. This may reduce the number of constants, e.g. $C_2 = C_4 = 0$. Then put the remaining equations into matrix form, and solve for the constants, either by row operations or by taking the determinant

Damping Problems

W is energy loss/cycle, V is peak potential energy (of whole system), η is the loss factor. Some formulas are:
$$\boxed{W = \int_0^{\frac{2\pi}{\omega}} f_d dx}$$
where, for a linear dashpot
$$\boxed{\textbf{Linear dashpot:} \quad f_d = c\dot{x}}$$
where x is the compression of the damper. This gives
$$W = \int_0^{\frac{2\pi}{\omega}} c\frac{dx}{dt}dx$$
$$\boxed{W = \int_0^{\frac{2\pi}{\omega}} c\left(\frac{dx}{dt}\right)^2 dt}$$
The loss factor is calculated as
$$\boxed{\eta = \frac{W}{2\pi V}}$$
To solve damping problems

1. Do lagrangian to get EOM and find the natural frequencies and mode shapes *assuming there is no damping*
2. Propose a solution of the form
$$\underline{x}(t) = \underline{a}\sin(\omega t)$$
3. Use modes to break this into components
4. Differentiate this solution to get $\frac{dx}{dt}$ and plug into the integral to evaluate W.

Rigid Symmetric Body EOM

$$\underline{\omega} = \omega_1\hat{\underline{e}}_1 + \omega_2\hat{\underline{e}}_2 + \omega_3\hat{\underline{e}}_3$$
$$\underline{H}_c = I_1\omega_1\hat{\underline{e}}_1 + I_2\omega_2\hat{\underline{e}}_2 + I_3\omega_3\hat{\underline{e}}_3$$
$$\underline{\tau}^{\text{ext}} = \frac{d}{dt}\underline{H}_c$$
$$\begin{aligned}\frac{d}{dt}\underline{H}_c &= I_1(\dot{\omega}_1\hat{\underline{e}}_1 - \omega_1\omega_2\hat{\underline{e}}_3 + \omega_1\omega_3\hat{\underline{e}}_2) \\ &+ I_2(\dot{\omega}_2\hat{\underline{e}}_2 + \omega_2\omega_1\hat{\underline{e}}_3 - \omega_2\omega_3\hat{\underline{e}}_1) \\ &+ I_3(\dot{\omega}_3\hat{\underline{e}}_3 - \omega_3\omega_1\hat{\underline{e}}_2 + \omega_3\omega_2\hat{\underline{e}}_1)\end{aligned}$$
Set this equal to $\underline{\tau}^{\text{ext}}$ and group components together to get EOM. Can simplify with symmetry, e.g. $I_1 = I_2$.
$$\tau_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)$$
$$\tau_2 = I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)$$
$$\tau_3 = I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)$$

Euler Angles

Start with coordinate system C_{XYZ}. Rotate ϕ about Z and get C_{abc} axes, then rotate θ about a and get C_{xyz}, and finally rotate ψ about z to get C_{123} axes, which are the body fixed axes.
$$\begin{aligned}\underline{\omega} &= \dot{\phi}\hat{\underline{e}}_Z + \dot{\theta}\hat{\underline{e}}_x + \dot{\psi}\hat{\underline{e}}_3 \\ &= \omega_1\hat{\underline{e}}_1 + \omega_2\hat{\underline{e}}_2 + \omega_3\hat{\underline{e}}_3\end{aligned}$$
where
$$\omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

Torque-Free Precession

When there is **no torque** acting on the system, angular momentum principle $\frac{d}{dt}\underline{H}_C = 0$ tells us that $\underline{H}_C$ **is constant**, and since it is a vector this means its magnitude and direction are constant. So, we can **choose the coordinate system C_{XYZ} such that the Z axis is aligned with $\underline{H}_C$**.
$$\begin{aligned}\underline{H}_C &= H_C\hat{\underline{e}}_Z \\ &= H_C(\sin\theta\hat{\underline{e}}_y + \cos\theta\hat{\underline{e}}_z) \\ &= H_C(\sin\theta\sin\psi\hat{\underline{e}}_1 + \sin\theta\cos\psi\hat{\underline{e}}_2 + \cos\theta\hat{\underline{e}}_3)\end{aligned}$$
Compare this expression for $\underline{H}_C$ to the earlier one
$$\underline{H}_C = I_1\omega_1\hat{\underline{e}}_1 + I_2\omega_2\hat{\underline{e}}_2 + I_3\omega_3\hat{\underline{e}}_3$$
Components have to match exactly. This gives
$$\dot{\phi} = H_C\left(\frac{\sin^2\psi}{I_1} + \frac{\cos^2\psi}{I_2}\right)$$
$$\dot{\theta} = H_C\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin\theta\cos\psi\sin\psi$$
$$\dot{\psi} = H_C\cos\theta\left(\frac{1}{I_3} - \frac{\sin^2\psi}{I_1} - \frac{\cos^2\psi}{I_2}\right)$$
If the body has **symmetry**, say **about 3 axis, then** $I_1 = I_2 = I$ and we can see from these expressions that $\dot{\theta} = \theta_s =$ constant, that $\dot{\phi} = \frac{H_C}{I} = \Omega =$ constant, and $\dot{\psi} = [(I - I_3)/I_3]\Omega\cos\theta_s =$ constant. **This is torque-free precession.**

Spinning Top with Euler Angles

- Use LaGrange to get EOM, neglecting $\underline{v}_C$
- Using Euler angles for $\underline{\omega}$ in Lagrange $\underline{\omega} = \dot{\phi}\hat{\underline{e}}_Z + \dot{\theta}\hat{\underline{e}}_x + \dot{\psi}\hat{\underline{e}}_3$ Describe in terms of C_{xyz} with $\hat{\underline{e}}_Z = \sin\theta\hat{\underline{e}}_z + \cos\theta\hat{\underline{e}}_y$ $\hat{\underline{e}}_3 = \hat{\underline{e}}_z$ $\underline{\omega} = \dot{\phi}\sin\theta\hat{\underline{e}}_z + \dot{\phi}\cos\theta\hat{\underline{e}}_y + \dot{\theta}\hat{\underline{e}}_x + \dot{\psi}\hat{\underline{e}}_z$
- Do Lagrange, identify from $\delta\psi$ equation $I_3\omega_z =$ constant $\omega_z \approx \dot{\psi}$

Torque-Free Motion of a Rigid Body

$$\begin{aligned}
\underline{\omega} &= \phi\underline{\hat{e}}_Z + \dot{\psi}\underline{\hat{e}}_z \\
&= \Omega\sin\theta_s\underline{\hat{e}}_y + (\dot{\psi} + \Omega\cos\theta_s)\underline{\hat{e}}_z \\
\underline{H}_C &= I\omega_y\underline{\hat{e}}_y + I_3\omega_z\underline{\hat{e}}_z \\
&= I\Omega\sin\theta_s\underline{\hat{e}}_y + I_3(\dot{\psi} + \Omega\cos\theta_s)\underline{\hat{e}}_z \\
&= I\Omega(\sin\theta_s\underline{\hat{e}}_y + \cos\theta_s\hat{e}_z) \\
&= I\Omega\underline{\hat{e}}_Z
\end{aligned}$$

If we evaluate $\frac{d}{dt}\underline{H}_C$ we can solve for the rate of change of the angular rates as

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2
\end{aligned}$$

We have with no torque that $\underline{H}_C = I_3(\dot{\psi} + \dot{\phi})$ is constant and $\underline{\omega} = (\dot{\psi} + \dot{\phi})\underline{\hat{e}}_z$. Choose $\underline{H}_C$ to be parallel to $\underline{\hat{e}}_Z$. Use Euler equations to examine stability of steady rotation. $\omega = \omega_3\underline{\hat{e}}_3$. So $\dot{\omega}_1$ and $\dot{\omega}_2$ are basically constant, giving

$$\ddot{\omega}_1 + \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2}\omega_3^2\omega_1 = 0$$

and this is stable when $(I_1 - I_3)(I_2 - I_3) > 0$ and unstable when $(I_1 - I_3)(I_2 - I_3) < 0$. So stable when I_3 is either a maximum or minimum moment of inertia.

General Math Stuff

Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Using this for sine and cosine, small angles

$$\begin{aligned}
\cos(x) &= 1 - \frac{1}{2}x^2 \\
\sin(x) &= x - \frac{x^3}{6}
\end{aligned}$$

Identities

$$\begin{aligned}
\sin(u \pm v) &= \sin u\cos v \pm \cos u\sin v \\
\cos(u \pm v) &= \cos u\cos v \mp \sin u\sin v \\
e^{ix} &= \cos x + i\sin x \\
\sinh(x) &= \frac{1}{2}(e^x - e^{-x}) \\
\cosh(x) &= \frac{1}{2}(e^x + e^{-x}) \\
\sin(x) &= \frac{1}{2i}(e^{ix} - e^{-ix}) \\
\cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix}) \\
\sinh(x) + \cosh(x) &= e^x \\
\frac{d}{dx}\sinh(x) &= \cosh(x) \\
\frac{d}{dx}\cosh(x) &= \sinh(x) \\
\sinh(0) = 0 &\qquad \cosh(0) = 1
\end{aligned}$$

Integrals

$$\begin{aligned}
\int \sin^2(ax)dx &= \frac{x}{2} - \frac{\sin(2ax)}{4a} \\
\int \cos^2(ax)dx &= \frac{x}{2} + \frac{\sin(2ax)}{4a}
\end{aligned}$$

Writing sum of sin and cos as sin with a phase shift

$$A_1\sin(\omega x) + A_2\cos(\omega x) = \sqrt{A_1^2 + A_2^2}\left(\underbrace{\frac{A_1}{\sqrt{A_1^2 + A_2^2}}}_{\cos\phi}\sin(\omega x) + \underbrace{\frac{A_2}{\sqrt{A_1^2 + A_2^2}}}_{\sin\phi}\cos(\omega x)\right)$$

Can check that $\sin^2\phi + \cos^2\phi = 1$. Then use identity $\sin(\omega x + \phi) = \sin(\omega x)\cos\phi + \cos(\omega x)\sin\phi$ giving

$$A_1\sin(\omega x) + A_2\cos(\omega x) = A_3\sin(\omega x + \phi)$$

$$\frac{\sin\phi}{\cos\phi} = \frac{A_2}{A_1} \Rightarrow \phi = \tan^{-1}\frac{A_2}{A_1} \qquad \text{and} \qquad A_3 = \sqrt{A_1^2 + A_2^2}$$

Wave Equation on String

This page gives an outline of the general procedure to **derive the equations of motion**, **propose a general solution**, and **solve for constants using boundary and initial conditions** (here we assume the boundary conditions are both ends fixed, and zero initial conditions) in order to get the **mode shapes** and **natural frequencies.**
Physical assumptions: *homogenous string ρA = constant, the string is perfectly elastic (no resistance to bending), the tension is way more than gravity, and string only vibrates perfectly up and down.*
Non dispersive waves: anything that obeys the wave equation, e.g. a string, the speed of wave propagation is constant and independent of frequency. All energy travels the same speed independent of frequency. $\frac{T}{\rho A}$ is the wave speed. $\sqrt{\frac{T}{\rho A}}$ is phase velocity. In beams non dispersive waves, the high frequency waves travel ahead of the lower frequency waves.

1. **Derive governing equation**
   (a) Momentum in x-direction gives $T(x)$ is constant
   (b) Do momentum in the y-direction
   (c) Use small angles: $\sin(\alpha + \frac{\partial \alpha}{\partial x}dx) = \alpha + \frac{\partial \alpha}{\partial x}dx$ and $\tan(\alpha) = \alpha$

   The governing equation is $\boxed{T\frac{\partial^2 y}{\partial x^2} = \rho A\frac{\partial^2 y}{\partial t^2}}$
2. **Propose a general separable solution** $\boxed{y(x,t) = a(x)f(t)}$
   (a) Rearrange the governing equation as $C^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ where $\boxed{C^2 = \frac{T}{\rho A}}$ and propose $f(t) = Ae^{i\omega_n t}$ giving $y(x,t) = a(x)Ae^{i\omega_n t}$ and plug in
   (b) The governing equation becomes $\boxed{C^2\frac{\partial^2 a}{\partial x^2} + \omega_n^2 a(x) = 0}$ *note: C is always positive*
   (c) Propose $a(x) = Be^{i\lambda x}$ and get $\omega_n = C\lambda$
   (d) The total solution is then $\boxed{y(x,t) = Be^{i\frac{\omega_n}{C}x}Ae^{i\omega_n t}}$ which can be decomposed into sine and cosine as $y(x,t) = (B_1\sin(\lambda x) + B_2\cos(\lambda x))(A_1\sin(\omega_n t) + A_2\cos(\omega_n t))$
3. **Apply boundary and initial conditions** to get the constants
   (a) Apply boundary conditions $y(x=0,t) = y(x=L,t) = 0$ gives $B_2 = 0$ and $B_1\sin(\frac{\omega_n}{C}L) = 0$ so $\frac{\omega_n}{C}L = n\pi$ where $n = 1,2,3\dots$. So $\omega_n = \frac{Cn\pi}{L}$. The solution becomes $y(x,t) = B_1\sin(\frac{\omega_n}{C}x)(A_1\sin(\omega_n t) + A_2\cos(\omega_n t))$
   (b) Apply initial conditions $y(x,t=0) = 0$ gives $A_2 = 0$ reducing solution to $y(x,t) = B_1\sin(\frac{\omega_n}{C}x)A_1\sin(\omega_n t)$ or by combining the constants $\boxed{y(x,t) = C_n\sin(\frac{\omega_n}{C}x)\sin(\omega_n t)}$
4. Now we have the governing equation, now we see if it is **self-adjoint** if it satisfies the following conditions

   i) $\boxed{\int u\rho A v dx = \int v\rho A u dx}$

   ii) $\boxed{\int v\left(-T\frac{\partial^2}{\partial x^2}\right)u dx = \int u\left(-T\frac{\partial^2}{\partial x^2}\right)v dx}$

   The first condition is satisfied automatically, since u and v (in our case $a(x)$ and $f(t)$ commute. We show that the second condition holds by doing integration by parts twice.

   $$\int_0^L \underbrace{a_i}_{u}\underbrace{\left(-T\frac{\partial^2}{\partial x^2}\right)a_j dx}_{dv} = \underbrace{a_i}_{u}\underbrace{\left(-T\frac{\partial}{\partial x}(a_j)\right)}_{v}\Big|_0^L + \int_0^L \underbrace{T\frac{\partial}{\partial x}(a_j)}_{v}\underbrace{\frac{da_i}{dx}dx}_{du}$$

   one more integration by parts

   $$\int_0^L \underbrace{\frac{da_i}{dx}}_{u}\underbrace{T\frac{\partial}{\partial x}(a_j)dx}_{dv} = \underbrace{\frac{da_i}{dx}}_{u}\underbrace{(-Ta_j)}_{v}\Big|_0^L + \int_0^L \underbrace{Ta_j}_{v}\underbrace{\frac{d^2a_i}{dx^2}dx}_{du}$$

   gives

   $$\int_0^L a_i\left(-T\frac{\partial^2}{\partial x^2}\right)a_j dx = a_i\left(-T\frac{\partial}{\partial x}(a_j)\right)\Big|_0^L - \left(\frac{da_i}{dx}(-Ta_j)\right)\Big|_0^L + \int_0^L a_j\left(-T\frac{\partial^2}{\partial x^2}(a_i)\right)dx$$

   and since we evaluate the first two terms on the right hand side at $x = 0$ and $x = L$, the boundary conditions dictate that $a_i = a_j = 0$ here, thus proving the system is **self-adjoint**. *Self-adjointness depends on the boundary conditions.*
5. Now we use the self adjoint property to show that the modes are **orthogonal**, where a_j and a_i are orthogonal functions if they satisfy

   $$\boxed{\int_0^L a_j a_i dx = 0}$$

   (a) Start with the **governing equation** for the **spatial function** for two different modes a_i and a_j, where $i \neq j$.

   $$C^2\frac{\partial^2 a}{\partial x^2} + \omega_n^2 a(x) = 0$$

   (b) Since the governing equation equals zero, we can post-multiply each of these expressions by the *other mode*, sum them, and it is still zero.

   $$\left(C^2\frac{\partial^2 a_i}{\partial x^2} + \omega_n^2 a_i(x)\right)a_j + \left(C^2\frac{\partial^2 a_j}{\partial x^2} + \omega_n^2 a_j(x)\right)a_i = 0$$

   (c) Expand this expression and **integrate from** 0 **to** L. Using the self-adjoint property which we just showed, we get

   $$\boxed{\frac{1}{C^2}\left(\omega_i^2 - \omega_j^2\right)\int_0^L a_i a_j dx = 0}$$

   (d) Since the natural frequencies corresponding to each of these modes is different, the integral must be zero, satisfying the definition and showing the modes are orthogonal.
6. To find the i**-th modal mass** and i**-th modal stiffness**, write down the governing *spatial* differential equation $T\frac{\partial^2 a}{\partial x^2} + \rho A\omega_n^2 a(x) = 0$ for a mode a_i and multiply both sides by a_i and dx, rearrange and integrate from 0 to L.

   $$a_i\left(T\frac{\partial^2 a_i}{\partial x^2}\right)dx + \rho A\omega_n^2 a_i^2(x)dx = 0$$

   giving the i-th modal mass and i-th modal stiffness as

   $$\omega_n^2\underbrace{\rho A\int_0^L a_i^2(x)dx}_{m_i\delta_{ij}} = \underbrace{\int_0^L a_i\left(-T\frac{\partial^2 a_i}{\partial x^2}\right)dx}_{k_i\delta_{ij}}$$

   And we can divide and solve for ω_R, which are regular modes? which gives the Rayleigh quotient.

   $$\omega_R^2 = \frac{\int_0^L a_i\left(-T\frac{\partial^2 a_i}{\partial x^2}\right)dx}{\rho A\int_0^L a_i^2(x)dx} = \frac{k_i\delta_{ij}}{m_i\delta_{ij}}$$

Modal Decomposition

When the string problem is forced, the governing equation is

$$\boxed{\rho A\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = f_0(x)\sin\Omega t}$$

We propose the same separable solution as before, where we make explicit that there are an infinite number of solutions, indexed by i, and the total solution is the sum

$$y(x,t) = \sum_i a_i(x)f_i(t)$$

Plugging this into the governing equation we get

$$\sum_i\left(\rho A a_i(x)\frac{\partial^2 f_i}{\partial t^2} - f_i(t)T\frac{\partial^2 a_i}{\partial x^2}\right) = f_0(x)\sin\Omega t$$

Left multiply by a_i and integrate across the beam from 0 to L

$$\sum_i\left(\int_0^L \rho A a_i^2(x)\frac{\partial^2 f_i}{\partial t^2}dx - \int_0^L a_i(x)T\frac{\partial^2 a_i}{\partial x^2}f_i(t)dx\right) = \int_0^L a_i f_0(x)\sin\Omega t dx$$

giving

$$\sum_i\left(\frac{\partial^2 f_i}{\partial t^2}\underbrace{\int_0^L \rho A a_i^2(x)dx}_{m_i\delta_{ij}} + f_i(t)\underbrace{\int_0^L -a_i(x)T\frac{\partial^2 a_i}{\partial x^2}dx}_{k_i\delta_{ij}}\right) = \int_0^L a_i f_0(x)\sin\Omega t dx$$

giving

$$\frac{\partial^2 f_i}{\partial t^2}m_i\delta_{ij} + k_i\delta_{ij}f_i(t) = \sin\Omega t\int_0^L a_i f_0(x)dx$$

Solving for $f_i(t)$

$$k_i f_i(t) = \sin\Omega t\int_0^L a_i f_0(x)dx - \frac{\partial^2 f_i}{\partial t^2}m_i$$

$$f_i(t) = \frac{\int_0^L a_i(x)f_0(x)dx}{k_i}\sin\Omega t - \frac{\frac{\partial^2 f_i}{\partial t^2}m_i}{k_i}$$

Boundary Conditions

Rollers at End with Spring

$$\boxed{T\frac{dy}{dx} = ky}$$

Euler-Bernoulli Beam Equation

Proposing Separable Solution

Given the governing PDE for a bending beam

$$\boxed{\rho A \frac{\partial^2 y}{\partial t^2} = -EI \frac{\partial^4 y}{\partial x^4}}$$

we propose a solution of the form

$$\boxed{y(x,t) = a(x)y(t)}$$

Evaluating the necessary derivatives given this solution form we get

$$\frac{\partial^2 y}{\partial t^2} = a(x)\frac{d^2 f}{dt^2} \quad \text{and} \quad \frac{\partial^4 y}{\partial x^4} = \frac{d^4 a}{dx^4} f(t)$$

substituting in

$$\rho A a(x) \frac{d^2 f}{dt^2} = -EI \frac{d^4 a}{dx^4} f(t)$$

which can be separated as

$$\rho A \frac{1}{f(t)} \frac{d^2 f}{dt^2} = -EI \frac{1}{a(x)} \frac{d^4 a}{dx^4} = \text{constant}$$

And so now we can solve each ODE separately now. This leads to a solution of the form

$$\boxed{y(x,t) = a(x) \sin \omega(t)}$$

Plugging this back into the governing equation, we reduce the equation to an ODE and then only have to find $a(x)$

$$\boxed{\frac{d^4 a}{dx^4} - \lambda^4 a(x) = 0} \quad \text{where} \quad \boxed{\lambda^4 = \frac{\rho A \omega^2}{EI}}$$

Self-Adjointness

Now we have the governing equation for beams, now we see if it is **self-adjoint** if it satisfies the following conditions

i) $\boxed{\int u \rho A v dx = \int v \rho A u dx}$

ii) $\boxed{EI \int u \frac{d^4 v}{dx^4} dx = EI \int v \frac{d^4 u}{dx^4} dx}$

where the integrals are evaluated from one end of the beam to the other, usually 0 to L. The first condition is trivial, and we can show the second condition holds by applying integration by parts. Additionally, from doing this integration, we also find the following relationship

$$\boxed{\int_0^L v \frac{d^4 u}{dx^4} dx = \int_0^L \frac{d^2 u}{dx^2} \frac{d^2 v}{dx^2} dx}$$

Orthogonality

To show **orthogonality** of the modes, start with the spatial governing equation for two modes a_i and a_j with $i \neq j$

$$\boxed{EI \frac{d^4 a_i}{dx^4} - \rho A \omega^2 a_i(x) = 0}$$

$$\boxed{EI \frac{d^4 a_j}{dx^4} - \rho A \omega^2 a_j(x) = 0}$$

Left multiply the first equation by a_j and the second by a_i. Integrate across the beam (from 0 to L) and subtract

$$EI \int_0^L a_j \frac{d^4 a_i}{dx^4} dx - \rho A \omega^2 \int_0^L a_j a_i dx - EI \int_0^L a_i \frac{d^4 a_j}{dx^4} dx + \rho A \omega^2 \int_0^L a_i a_j dx = 0$$

Use self-adjointness to cancel out terms, giving

$$\rho A \omega^2 \int_0^L a_i a_j dx = \rho A \omega^2 \int_0^L a_j a_i dx$$

Thus showing the modes are orthogonal.

Finding i-th Modal Mass and Stiffness

To find the i-th modal mass and stiffness, again use the spatial governing ODE for mode a_i

$$EI \frac{d^4 a_i}{dx^4} - \rho A \omega^2 a_i(x) = 0$$

left multiply by a_i, and integrate across the beam from 0 to L

$$EI \int_0^L a_i \frac{d^4 a_i}{dx^4} dx - \rho A \omega^2 \int_0^L a_i^2(x) dx = 0$$

use the additional property from self-adjointness to replace the fourth derivative as the product of two second derivatives

$$\underbrace{EI \int_0^L \left(\frac{d^2 a_i}{dx^2}\right)^2 dx}_{k_i \delta_{ij}} - \omega^2 \underbrace{\rho A \int_0^L a_i^2(x) dx}_{m_i \delta_{ij}} = 0$$

where k_i and m_i are the i-th modal stiffness and mass, respectively. From this we can find ω as

$$\omega^2 = \frac{k_i \delta_{ij}}{m_i \delta_{ij}}$$

Boundary Conditions

Free End **No moment, no shear.**

Fixed or Clamped End **Displacement and slope are zero.**

Pinned End **No displacement, no moment.** Remember $M_b = EI \frac{\partial^2 y}{\partial x^2}$ so for pinned end this means the second derivative must be zero.

Point Mass at End **No moment**, external shear due to mass boundary condition, from conservation of linear momentum in y-direction.

$$\boxed{-Q = m \frac{\partial^2 y}{\partial t^2}}$$

Applied moment From $M_b = EI \frac{\partial^2 y}{\partial x^2}$, the boundary condition due to M_{applied} is

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{EI} M_{\text{applied}}}$$

Forced Beam Response

To solve the forced response, always solve the unforced problem first.

$$y(x,t) = \sum_i a_i(x) f_i(t)$$

$$\sum_i \left(EI \frac{d^4 a_i}{dx^4} f_i + \rho A a_i \frac{d^2 f_i}{dt^2} \right) = f_0 \sin(\Omega t)$$

$$\int_0^L a_j \left[\sum_i \left(EI \frac{d^4}{dx^4} f_i + \rho A a_i \frac{d^2 f_i}{dt^2} \right) = f_0 \sin(\Omega t) \right] dx$$

using orthogonality of modes (ith modal mass and stiffness?)

$$k_j f_j + m_j \frac{d^2 f_j}{dt^2} = \int_0^L a_j f_0 \sin(\Omega t) dx$$

The solution is

$$f_j(t) = \frac{\int_0^L a_j(x) dx}{m_j(\omega_j^2 - \Omega)} f_0 \sin(\Omega t) + C_j \sin(\omega_j t + B_j)$$

Find C_j and B_j by applying initial conditions.